Le Morpion Solitaire

Achim Flammenkamp, Departments of Mathematics, University Bielefeld

created 2003–Mar–19, revised 2008-Feb–08, updated 2011–Nov–10

There is a common known solitaire game in France: le Morpion. It is played on a checkered piece of paper with a pencil. At the beginning of the game the paper contains a so called Maltesian Cross consisting of 36 marked points on its grid. Your aim is to mark as many as possible further points on this grid. A new point can only be marked, if either a straight line segment of length five in one of the two diagonal directions or the two orthogonal directions relative to the grid could be chosen to cover this new point, provided then this line segment is completely covered by marked points and at most one single marked point of it is already covered by other line segments.

We like to put these conditions in a precise mathematical model: Given the grid $\mathbb{Z}^2$, then we denote by P_0 the special start configuration

$$\{(x,y) \in \mathbb{Z}^2 :\ 0 \leq x, y \leq 9,\ \lambda(x,y)\}$$

where λ is the characteristic boolean function of the start configuration which is defined as

$$\lambda(x,y) = \begin{cases} 3 \nmid x\ , 3 \nmid y : & \text{FALSE} \\ 3 \nmid x\ , 3 \mid y : & 2 \mid \lfloor \frac{x}{3} \rfloor \neq 3 \mid \lfloor \frac{y}{3} \rfloor \\ 3 \mid x\ \ , 3 \nmid y : & 3 \mid \lfloor \frac{x}{3} \rfloor \neq 2 \mid \lfloor \frac{y}{3} \rfloor \\ 3 \mid x\ \ , 3 \mid y : & 3 \nmid \lfloor \frac{x}{3} \rfloor \vee 3 \nmid \lfloor \frac{y}{3} \rfloor \end{cases}$$

using the binary boolean divide-operator $\mid$. Then we have $r := |P_0| = 36$. This configuration P_0 has all the (movement-)symmetries of the grid $\mathbb{Z}^2$.

```
9  .  .  .  o  o  o  o  .  .  .
8  .  .  .  o  .  .  o  .  .  .
7  .  .  .  o  .  .  o  .  .  .
6  o  o  o  o  .  .  o  o  o  o
5  o  .  .  .  .  .  .  .  .  o
4  o  .  .  .  .  .  .  .  .  o
3  o  o  o  o  .  .  o  o  o  o
2  .  .  .  o  .  .  o  .  .  .
1  .  .  .  o  .  .  o  .  .  .
0  .  .  .  o  o  o  o  .  .  .
   0  1  2  3  4  5  6  7  8  9
```

Another short description of the set is to consider it as generated by a "walk" on the grid which consists of 12 steps. Each step "adds" 3 further points to the path and is either a "turn left" denoted by $-$ or a "turn right" denoted by $+$. Then the total path, i.e. the set P_0, can be coded as the sequence $+--+--+--+--$ or any of its cyclic shifts (because the walk describes a closed path) or its inversion (because "left" and "right" are arbitrary defined). Where on the grid we start our walk, and what the start direction of the four possibilities $(1,0),(0,1),(-1,0),(0,-1)$ is changes only the position of the set relative to the coordinate origin, not its geometric form.

Next we need the set $\mathcal{L}$ of all line segments of length five to be defined as

$$\{\ ((x-hd, y-ht))_{h\in\{-2,-1,0,1,2\}} :\ x, y \in \mathbb{Z}\ \ (d,t) \in \{(1,0),(0,1),(1,-1),(1,1)\}\ \}$$

.

Now we search for an as long as possible sequence $((p_i,\ell_i))_{i\in\{1,2,\ldots,n\}}$ of pairs of points $p_i \in \mathbb{Z}^2$ and line segments $\ell_i \in \mathcal{L}$ satisfying the following three conditions $\forall\, i \in \{1,2,\ldots,n\}$:

i $p_i \notin P_0\,,\ \forall\, j < i :\ p_i \neq p_j$

ii $p_i \in \ell_i\,,\ \forall\, j < i :\ |\ell_i \cap \ell_j| \leq 1$

iii $\exists\, i_1, i_2, i_3, i_4, i_5 \in \{1,2,\ldots,i\} :\ \ell_i = (p_{i_1}, p_{i_2}, p_{i_3}, p_{i_4}, p_{i_5})$

Condition (i) means: add each move a further, previously unmarked point to the grid. Condition (ii) means: this point must be covered by a new line segment, but the new line segment is not allowed to have more than one point in common with any other line segment. and condition (iii) says that the new line segment must be covered completely by marked points.

An open problem is still to determine the maximal possible value of n. In this paper we like to show that such an n exists indeed and we give an upper bound on the maximum n.

For this purpose we define for every $k \leq n$ an evaluation function f_k on $\mathbb{Z}^2$ depending on the set $P_k := P_0 \cup \{p_i\}_{i\in\{1,2,\ldots,k\}}$ and the set $\{\ell_i\}_{i\in\{1,2,\ldots,k\}}$ of the points and line segments of our sequence $((p_i,\ell_i))_{i\in\{1,2,\ldots,n\}}$. To do this we need the for all $\ell \in \mathcal{L}$ the notation of

$$\ell^* := (p_2, p_3, p_4) \quad \text{if} \quad \ell = (p_1, p_2, p_3, p_4, p_5)$$

which is simply the inner part, consisting of 3 points, of any line segment.

Now we define for every $q \in \mathbb{Z}^2$, every $k \leq n$ and each $((p_i,\ell_i))_{i\in\{1,2,\ldots,n\}}$

$$\begin{aligned} f_k(q) \;&=\; 8\delta_{q\in P_k} - \sum_{i\in\{1,2,\ldots,k\}} \delta_{q\in\ell_i} - \sum_{i\in\{1,2,\ldots,k\}} \delta_{q\in\ell_i^*} \\ &\text{with} \qquad \delta_{\mathrm{TRUE}} = 1 \ \text{ and } \ \delta_{\mathrm{FALSE}} = 0\,. \end{aligned} \tag{1}$$

This implies that holds

$$\forall\, k \in \{1, 2, \ldots, n\}\; \forall\, q \in \mathbb{Z}^2 : \quad 0 \leq f_k(q) \leq 8$$

where the lower bound is due to the fact that the line segments have 4 possible directions and for each direction there is either at most one full cover by the middle of a line segment or two half covers by line ends. Moreover, the fact that for all sequencs $((p_i, \ell_i))_{i\in\{1,2,\ldots,k\}}$ each point p_i adds 8 and its corresponding line ℓ_i decreases certain f_k by $5+3$ in the summation of (1), we get — summing f_k up over the whole grid —

$$\forall\, k : \quad \sum_{q\in\mathbb{Z}^2} f_k(q) = \sum_{q\in P_0} 8 = 8r$$

which is an invariante for each k and any sequence.

Given a fixed k, then a point $q \in P_k$ for whichs holds $f_k(q) = 0$ must be saturated in all 4 directions and must be an inner point of the set P_k. Thus the set of points with $f_k > 0$ must contain at least the border points of the configuration of P_k because these can't be saturated completed in all 4 directions by line segments. Of course this holds for all $k' > k$ provided $P_{k'}$ exists because $f_k(q)$ is a monotonic decreasing function in k for fixed $q \in P_k$.

Therefore the set $P_k \subset \mathbb{Z}^2$ should have a border measure as low as possible to maximize n. This leads immediately to a simple upper bound: Giving $8r$-many border points for a subset S the maximization of $|S|$ means S must be at least a convex set and because it is a subset in $\mathbb{Z}^2$ which points are created by line segments in only 4 possible directions it must be a convex $2\cdot 4$ -gon, namely an octagon. Trivially an octagon has 8 edges, so the full-symmetric octagon with side length $r+1$ has exactly $8r$ border points and a total number of $7r^2 + 4r + 1$ points. And with $r = 36$ this gives our first upper bound of 9217 for $P_n \subset |S|$. A border point of a convex set can be covered only in at most 5 directions of the 8 possible directions under consideration in the grid $\mathbb{Z}^2$. Thus we notice that even $f_k(q) \geq 3$ must hold for any border point of P_n. Therefore we have at most $\frac{8r}{3}$ border points and a total of $7\frac{r^2}{9} + 4\frac{r}{3} + 1 = 1057$ points for the full-symmetric octagon with side length $\frac{r}{3} + 1$.

This rough estimation can be improved by refinement, still assuming we can cover the inner points of the optimal convex octagon perfectly, such that only near the border of the octagon $f > 0$. In detail look at a typical corner of such an octagon:

$$\begin{array}{ccccccc}
. & . & . & . & . & . & . \\
. & . & . & . & . & . & . \\
. & . & . & 4 & 3 & 3 & 3 \\
. & . & 3 & 1 & 0 & 0 & 0 \\
. & 3 & 1 & 0 & 0 & 0 & 0 \\
3 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

Here a dot . indicates a point $q \notin P$, having $f(q) = 0$, else the value $f(p_i)$ is shown. Assuming the horizontal and vertical diameters of the octagon to be b_1 and b_2 and the four corner cut-off-sizes to be $a_1, a_2, a_3, a_4 \leq \min\{b_1, b_2\}$, we get the condition

$$3(2b_1 + 2b_2) - 2(a_1 + a_2 + a_3 + a_4) - 4 \leq 8r \tag{2}$$

for the border weight.

Under this condition we have to maximize

$$b_1 b_2 - \sum_{j \in \{1,2,3,4\}} \frac{a_j(a_j + 1)}{2} \tag{3}$$

the total number of points in respect to $b_1, b_2, a_1, a_2, a_3, a_4$. Without loss of generality we may assume that $b_1 \leq b_2$ and $a_1 \leq a_2 \leq a_3 \leq a_4$. Because of the special form of the restriction and the objective function we know that for the maximum must hold $b_1 \leq b_2 \leq b_1 + 1$ and $a_1 \leq a_4 \leq a_1 + 1$. So, renaming a_1 to a and b_1 to b, we rewrite the restriction to

$$12b + 6\beta - 8a - 2\sigma - 4 \leq 8r$$

with $b_2 - b_1 = \beta \in \{0, 1\}$ and $\sum_j a_j - a = \sigma \in \{0, 1, 2, 3\}$ and have to maximize

$$b^2 + \beta b - (a + 1)(2a + \sigma)$$

in respect to b, β, a, σ. Because σ and a can be used to get equality in the restriction while at most increasing the objective function, we get with the "active" restriction

$$a = a(b, \beta) = \lfloor \frac{6b + 3\beta - 2 - 4r}{4} \rfloor \geq 0$$

and

$$\sigma = 6b + 3\beta - 2 - 4r - 4a \quad .$$

Substituting a and σ in the objective function, keeping in mind that $6b + 3\beta \geq 4r + 2$ must hold, we yield

$$\max_{b,\beta} \quad b^2 + \beta b - (a(b, \beta) + 1)(6b + 3\beta - 4r - 2 - 2a(b, \beta))$$

For $r = 36$, the maximum value 741 is attained with $b = 31$ and $\beta = 0$, implying $a = 10$ and $\sigma = 0$. Because $\sigma = \beta = 0$ holds this octagon must be rotational symmetric in $\mathbb{Z}^2$ and moreover $b - 2a = a + 1$ means it is even a full symmetric octagon with side length $a + 1$ consisting of $7a^2 + 4a + 1$ points.

Remark: If we would drop the requirement that a_1, a_2, a_3, a_4 and b_1 and b_2 must be integers, we could approximate the solution by knowing that in the maximum must hold $a_1 = a_2 = a_3 = a_4$, $b_1 = b_2$ and the restriction

$6(2b) - 8a - 4 \leq 8r$ must hold with equality. Putting this together, we have to maximize the term $b^2 - \frac{1}{2}((3b-2r)^2 - 1)$ in b, which means that $2b - (3b-2r)3 = 0$ must hold. Then we would get for the optimal b the value $\frac{6r}{7}$ and the maximal value $\frac{4r^2}{7} + \frac{1}{2}$, resulting in the case $r = 36$ in a maximal value of $741.0714\ldots$ as good as our bound on n in the exact all integer estimation.

To get a better bound we like to estimate the size of P_n better. The main idea is to use the alignment restriction of the line segments (of length 5). In our model each point q has "base-costs" of 8 which can be decreased by line segment covers. Therefore our aim is to cover as many as possible points with four lines chosen from the four possible directions to push $f_k(q)$ down to 0 for each marked point. Denote the horizontal diameter, this is the maximal number of lines orthogonal to the x-axis to cover P_n, again by b_1 and the vertical diameter, the maximal number of parallel lines to the x-axis to cover P_n, with b_2. Similar let the diameter in the two diagonal directions be d_1 and d_2. Then we get a to (3) equivalent formula for the number of points covered in all four directions to be

$$b_1 b_2 - \lfloor \frac{(b_1 + b_2 - d_1)^2}{4} \rfloor - \lfloor \frac{(b_1 + b_2 - d_2)^2}{4} \rfloor \tag{4}$$

Any $m+1$ successive line segments lying on the same line overlap only at their end points and thus cover $4(m+1)+1$ points, causing a waste of 2 at the end points for each such line regarding the global $\sum_q f_k(q)$. Furthermore because of this alignment restriction modulo 4, we must have an average excess of $\frac{1}{4}\sum_{i=0}^{3} i = \frac{3}{2}$ of uncovered points for each independent line direction. Because any 2 (but no 3) of the 4 directions are independent, we get at least for the lines in the other 2 directions these costs for the miss-aligment. Each point in this excess, which can't be covered by the line gives a further costs of 2. Trying to avoid mis-alignment on the first and last lines of each set of parallel line-covers save at most 3/2 for each set, but on the other hand we have to pay $2 \cdot 2$ costs due to the fact that no line of length 1 is availible for each such set of paralle line-covers which out-weights this small win – so we neglect this small effect. Because $d_1, d_2 \geq \max\{b_1, b_2\}$ we can put all together to the condition

$$2(b_1 + b_2 + d_1 + d_2) + 2(\alpha b_1 + (3-\alpha) b_2) \leq 8r \qquad 0 \leq \alpha \leq 3 \tag{5}$$

Maximizing (4) under the asssumption $b_1 \leq b_2$, results in the non-integer case in the unique solution

$$b_1 = \frac{3}{8}r \quad b_2 = \frac{6}{8}r \quad d_1 = d_2 = \frac{7}{8}r \quad \alpha = 3$$

which give the maximal value of at most $\frac{r^2+2}{4}$. The all integer solution for our case $r = 36$, give a maximal value of 324 with the two different solutions $b_1 = 14, b_2 = 26, d_1 = d_2 = 31, \alpha = 3$ and $b_1 = 13, b_2 = 28, d_1 = d_2 = 32, \alpha =$

3 — for each solution (5) is active. Remark: we had assumed that P_n is a convex n-gon else (4) would be invalid and thoughts like “average excess” undefined.

The current, in November 2011, record of $n = 178$ of a morpion-sequence was set by Christopher D. Rosin in August 2011. Because always $p_i \in \ell_i$ we may give $\ell_i - p_i$ inplace of ℓ_i. To get an even more compact list of these pairs, the line segment can be coded unique as a difference Δ_i of its center point from p_i and its direction. The four possible directions $(1, 0)$, $(0, 1)$, $(1, -1)$ and $(1, 1)$ are representated as $-$, $|$, $/$ and $\backslash$ respectively.

Now the sequence of pairs p_i, Δ_i for this record configuration follows:

i	p_i	Δ_i	i	p_i	Δ_i	i	p_i	Δ_i	i	p_i	Δ_i
1	(3,-1)	2|	2	(4,3)	-2−	3	(2,0)	2−	4	(2,7)	0\
5	(5,1)	0\	6	(5,3)	1−	7	(0,2)	2|	8	(6,-1)	2|
9	(4,1)	0/	10	(2,1)	2−	11	(7,9)	-2−	12	(2,2)	0/
13	(2,4)	-2|	14	(4,6)	-2−	15	(3,5)	-1\	16	(3,4)	1|
17	(5,6)	1−	18	(4,5)	0\	19	(5,4)	-1/	20	(6,5)	-2\
21	(6,4)	1|	22	(4,2)	0\	23	(4,4)	0|	24	(7,-1)	-2/
25	(1,4)	1−	26	(1,2)	1−	27	(4,-1)	-2/	28	(5,-1)	0−
29	(7,2)	-1\	30	(5,2)	-1|	31	(7,4)	-2\	32	(8,2)	-2−
33	(4,-2)	2|	34	(7,1)	-1\	35	(8,0)	-2/	36	(7,0)	1|
37	(8,-1)	-2/	38	(1,1)	1/	39	(1,5)	-2|	40	(4,8)	-2\
41	(8,1)	0|	42	(8,4)	-2−	43	(-1,7)	2/	44	(5,5)	-2\
45	(5,7)	-2|	46	(9,1)	-2/	47	(8,10)	-2\	48	(7,5)	-2/
49	(8,5)	-2−	50	(8,7)	-2|	51	(7,7)	-2|	52	(4,7)	1−
53	(2,9)	2/	54	(2,5)	0\	55	(-1,5)	2−	56	(2,8)	-2|
57	(1,7)	0\	58	(0,7)	1−	59	(-1,8)	2/	60	(10,1)	-2−
61	(9,2)	-1/	62	(10,3)	-2\	63	(11,2)	-2/	64	(9,0)	2|
65	(10,-1)	-2/	66	(10,0)	-2−	67	(12,3)	-2\	68	(11,3)	-1−
69	(10,2)	-1|	70	(11,1)	-2/	71	(12,4)	-2\	72	(12,2)	-2−
73	(10,4)	0/	74	(11,4)	-1−	75	(11,5)	-2|	76	(12,6)	-2\
77	(10,5)	0/	78	(12,5)	-2−	79	(12,7)	-2|	80	(11,6)	-1\
81	(10,6)	0−	82	(10,7)	-2|	83	(11,8)	-2\	84	(11,7)	-2\
85	(9,7)	0−	86	(10,8)	-2\	87	(11,9)	-2|	88	(8,8)	2/
89	(9,8)	-2|	90	(10,9)	-2\	91	(7,8)	2−	92	(5,8)	0−
93	(3,10)	2/	94	(3,11)	-2|	95	(4,10)	2/	96	(4,11)	-2|
97	(5,10)	1/	98	(5,11)	-2|	99	(8,9)	2/	100	(9,9)	0−
101	(10,10)	-2\	102	(10,11)	-2|	103	(9,10)	-1\	104	(8,11)	2/
105	(7,10)	-1\	106	(6,10)	-1−	107	(11,10)	-2−	108	(7,11)	-2|
109	(8,12)	-2\	110	(8,13)	-2|	111	(6,12)	2/	112	(6,11)	-2|
113	(9,11)	-1−	114	(7,12)	-1\	115	(9,12)	-2|	116	(5,12)	2−
117	(4,13)	2/	118	(7,14)	2/	119	(6,13)	-1\	120	(7,13)	-2\
121	(7,15)	-2|	122	(5,13)	1−	123	(2,11)	2−	124	(4,12)	2/
125	(6,14)	-1\	126	(6,15)	-2|	127	(5,15)	2/	128	(5,14)	-1|
129	(3,12)	1\	130	(4,15)	2/	131	(3,15)	2−	132	(4,14)	-1|
133	(3,14)	2−	134	(3,13)	0|	135	(2,16)	2/	136	(1,9)	2/
137	(1,8)	-1|	138	(0,8)	1−	139	(2,10)	-1\	140	(-1,9)	2/
141	(0,9)	1−	142	(0,10)	-2|	143	(-1,6)	2\	144	(-2,7)	2/
145	(1,10)	-1\	146	(-1,10)	2−	147	(-1,11)	-2|	148	(-2,11)	2/
149	(2,12)	-2|	150	(1,11)	2\	151	(1,12)	2−	152	(1,13)	-2|
153	(0,14)	2/	154	(0,11)	0−	155	(2,13)	-1\	156	(0,13)	2−
157	(-1,14)	2/	158	(0,12)	0|	159	(2,14)	-1\	160	(-1,13)	2/
161	(2,15)	-1|	162	(1,14)	0−	163	(-1,12)	1\	164	(0,15)	2/
165	(-1,15)	-2|	166	(1,15)	0−	167	(0,16)	2/	168	(-2,12)	2\
169	(-3,13)	2/	170	(-3,12)	2−	171	(-2,13)	2/	172	(1,16)	-2\
173	(0,17)	2/	174	(0,18)	-2|	175	(-4,13)	2−	176	(1,17)	-2|
177	(-2,14)	1\	178	(-2,10)	2|						

Remark: In our solitaire game we are allowed to cover with lines pointing in 4 different directions and with line segments of length 5. If the line segments are shorter than 5, we would generate new points faster than we can exceed line segments which may enable us to generate infinite sized patterns for certain start configurations. On the other hand, if the line segments are longer than 5, we would run out quickly of new line segments availible to create a new point and cover the point-tuple. Its corresponding sum

$\sum_{q\in\mathbb{Z}^2} f'_k(q)$ would be a decreasing function in k becoming negative for k big enough, meaning impossible configurations these sizes. Thus the length $(4+1)$ is well-balanced to the number 4 of allowed directions for covering line segments.

Border point classification:

$$\begin{array}{cccccccccccccccc}
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & 5 & 3 & 3 & 4 & . & . & 6 & 4 & 3 & 3 & 3 & 3 \\
. & . & . & . & 3 & 0 & 0 & 1 & 4 & . & . & 3 & 1 & 0 & 0 & 0 \\
. & . & . & : & 2 & 0 & 0 & 0 & 3 & . & . & . & 3 & 1 & 0 & 0 \\
. & . & . & 3 & 1 & 0 & 0 & 0 & 3 & . & . & . & . & 3 & 1 & 0 \\
. & . & 3 & 1 & 0 & 0 & 0 & 0 & 2 & : & . & . & . & : & 1 & 0 \\
. & 5 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 3 & 3 & 2 & 1 & 0 \\
. & . & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

Here a dot . or a colon : indicates a point $q \notin P$, having $f(q)=0$, else the value $f(p_i)$ is shown. A : indicates a concave corner in the grid. We see, that "sharp" vertices has value 5 or 6, and 4 is also a convex corner. 3 is along the edge, a 180-corner and neighboured points with 1 and 2 indicate a concave vertex. Adding in a concave corner a further point, will never increase the total value of $\sum_{q\in\mathbb{Z}^2} f_k(q)$, but at every edge or convex corner it will always! Thus we may cover our (possible non-convex) set P_k always by a convex octagon S without increasing its invariant value. of the geometry of an octagon. Next follows a list of figures of all possible octagon vertices showing its local neigbourhood and indicating their f-value with the "vertex point" value indicated in bold:

convex vertices:

$$\begin{array}{ccc} . & . & . \\ . & \mathbf{7} & 6 \\ . & . & . \end{array} \qquad
\begin{array}{ccc} . & . & 6 \\ . & \mathbf{7} & . \\ . & . & . \end{array} \qquad
\begin{array}{ccc} . & . & . \\ . & \mathbf{6} & 4 \\ . & . & 3 \end{array} \qquad
\begin{array}{ccc} . & . & . \\ . & \mathbf{5} & 3 \\ . & 3 & 0 \end{array} \qquad
\begin{array}{ccc} . & . & 3 \\ . & \mathbf{5} & 2 \\ . & . & 3 \end{array} \qquad
\begin{array}{ccc} . & . & 3 \\ . & \mathbf{4} & 1 \\ . & 3 & 0 \end{array}$$

"flat" vertices:

$$\begin{array}{ccc} . & 3 & 0 \\ . & \mathbf{3} & 0 \\ . & 3 & 0 \end{array} \qquad
\begin{array}{ccc} . & . & 3 \\ . & \mathbf{3} & 1 \\ 3 & 1 & 0 \end{array}$$

concave vertices:

$$\begin{array}{ccccc} 3 & . & . & . & . \\ 1 & 3 & . & . & . \\ 0 & 1 & \mathbf{2} & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array} \qquad
\begin{array}{cccc} . & . & 3 & 0 \\ . & . & 2 & 0 \\ 3 & 2 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 \end{array} \qquad
\begin{array}{ccccc} 3 & . & . & . & 3 \\ 1 & 3 & . & 3 & 1 \\ 0 & 1 & \mathbf{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \qquad
\begin{array}{ccccc} 0 & 3 & . & . & 3 \\ 0 & 2 & . & 3 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & \mathbf{0} & 0 & 0 & 0 \end{array}$$

$$\begin{array}{ccccc} 0 & 3 & . & 3 & 0 \\ 0 & 3 & . & 3 & 0 \\ 0 & 2 & . & 2 & 0 \\ 0 & 1 & \mathbf{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \qquad
\begin{array}{ccccc} . & 2 & 1 & 0 & 0 \\ 2 & . & 2 & 1 & 0 \\ 1 & 2 & . & 1 & 0 \\ 0 & 1 & 1 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}$$

vertices description form left to right and top to bottom:

1. convex: degenerated orthogonal top
2. convex: degenerated diagonal top
3. convex: $\frac{\pi}{4}$-radian vertex
4. convex: $\frac{\pi}{2}$-radian orthogonal vertex
5. convex: $\frac{\pi}{2}$-radian diagonal vertex
6. convex: $\frac{3\pi}{4}$-radian vertex
7. flat: orthogonal edge
8. flat: diagonal edge
9. concave: $\frac{5\pi}{4}$-radian vertex
10. concave: $\frac{3\pi}{2}$-radian orthogonal vertex
11. concave: $\frac{3\pi}{2}$-radian diagonal vertex
12. concave: $\frac{7\pi}{4}$-radian vertex
13. concave: degenerated orthogonal hole
14. concave: degenerated diagonal hole

Now we could generate all convex triangle, rectangle, 5-gons, 6-gons, 7-gons and octagons with a given diameter in the directions horizontal, vertical and the two diagonally, call these values as before b_1, b_2, d_1, d_2. All their border vertices have angles of $\frac{\pi}{4}$, $\frac{\pi}{2}$ or $\frac{3\pi}{4}$.

Remark: Lets generallize the important function (1) for any set $S \subset \mathbb{Z}^2$ with the definitions:

$$f(S) = \sum_{q \in S} f_S(q) \qquad \text{and} \qquad \forall\, q \in S: \quad f_S(q) \;=\; 8 - \sum_{\substack{|q-p|=1 \\ p \in \mathbb{Z}^2}} \delta_{p \in S}$$

Any solution to the morpion-solitaire game can be covered by its octagonal hull. This unique octagon P has two important attributes: firstly it is a convex octagon and secondly $f(P) \leq f(S)$ where S denotes the covered set of points of the given solution. The last attribute is due to the fact, that adding a point $q \in P, q \notin S$ in a concave corner of a solution set S, we have $f_P(p) = f_S(p) - 1$ for at least four different points $p \in S$ but $f_P(q) \leq 4$.

The constructed largest convex octagon with $f(P) = 8r = 288$ has diameter 31 and $|P| = 741$, thus gives an upper limit of $n = |P| - r = 705$. But we can place at most 167 horizontal, 167 vertical and two-times 159 diagonal line segments of length 5 in it. Hence a total of 652 line segments for this full-symmetric octagon. There are 22323262 convex octagons with diameter $\leq 48 = \frac{1}{2}\frac{8r}{3}$. Of these the unique octagon with sidelengthes $12, 10, 13, 9, 14, 9, 13, 10$ which covers 738 points contains the most number of valid line segments of length 5, namely 665 and satisfies $2l \leq 8r$ where l denotes the total number of lines. See the remark "a waste of 2" in the sentence following equation (4). Hence 665 is an upper bound for any achievable morpion-solution.

Next follows a short table of the maximal number of possible moves $n(r)$ for an appropriate chosen start configuration with given small size r:

r	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$n(r)$	0	0	0	1	1	1	2	2	3	4	6	8	12	19	24	35

The values $n(r)$ with $r \geq 10$ were determined by complete enumeration of all (connected) start configurations of size r. A record configuration for $r = 16$ – where a number i indicates the position of p_i and o denotes a point in P_0 – is e.g.

.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
.	.	6	1	o	o	o	o	o	o	5	.	.	.	.
.	.	o	o	o	7	9	15	22	25	26	.	.	.	.
.	.	10	o	2	11	13	20	24	27	29	.	.	.	.
.	.	8	o	o	12	16	18	23	28	31	33	.	.	.
.	3	o	o	o	o	14	17	19	21	30	32	34	35	.
.	.	.	.	.	.	4	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

with its sequence $(\Delta_i)_{1\leq i\leq 35} =$ (-2|, 0|, 2−, 2\, -2−, 2−, -1/, 1/, -2−, 0|, 0/, 1|, -2−, 2\, -1/, 0|, 2\, -2−, 2\, 0|, -2−, -1/, 1\, 0|, -1/, -2−, -1/, 1|, -2−, 2\, 1|, 2\, -2−, 2\, -2−) . Remarkably, the pattern of new chosen points "walks" from left to right when proceeding from move 11 to move 30. This could be continued at infinity if a horizontal beam of points, which starts at the point 6 and proceeds rightwards to infinity, already would has been given in the start configuration.